

Eikonalization and unitarity constraints

P. Desgrolard^{1,a}, M. Giffon^{1,b}, E. Martynov^{2,c}, and E. Predazzi^{3,d}

¹ Institut de Physique Nucléaire de Lyon, IN2P3-CNRS et Université Claude Bernard, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France

² Bogoliubov Institute for Theoretical Physics, NAS of Ukraine, Metrologicheskaja 14b, 03143, Kiev-143, Ukraine

³ Dipartimento di Fisica Teorica, Università di Torino and Sezione INFN di Torino, Italy

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Abstract. An extensive generalization of the ordinary and quasi-eikonal methods is presented for the pp and $\bar{p}p$ elastic scattering amplitudes, which takes into account in a phenomenological way all intermediate multiparticle states involving the crossing even and crossing odd combinations of Reggeons. The formalism in this version involves a maximum of three parameters corresponding to the intermediate states which are possible in this configuration. The unitarity restriction is investigated and particular cases are discussed. An interesting result that emerges concerns the Odderon trajectory intercept: we find that unitarity dictates that this quantity *must* be below or equal unity unless a very peculiar equality exists between the coupling of the particles to the Pomeron and the Odderon.

1 Introduction

In phenomenological models for the elastic scattering of hadrons at high energy, based on perturbative QCD, the Pomeron is a simple pole in the complex angular momenta plane, lying at $t = 0$ on the right of $j = 1$. This means that the Pomeron trajectory has an intercept $\alpha_P(0) = 1 + \delta_P$ with $\delta_P > 0$. In this case, the Pomeron contribution to the asymptotic total cross section,

$$\sigma_P^{\text{tot}}(s) \propto (s/s_0)^{\delta_P}, \quad s_0 = 1 \text{ GeV}^2,$$

results in a violation of unitarity, due to the restriction set by the Froissart–Martin bound [1]

$$\sigma^{\text{tot}}(s) \leq C \ln^2(s/s_0) \quad (C \simeq 60 \text{ mb}).$$

Such a Pomeron – sometimes denoted as *supercritical*¹ – can only be considered as an input or a *Born Pomeron* and must be unitarized. To this aim, the eikonal method [2] and its generalizations are most often used [3–5].

While the eikonalization procedure is quite standard for one Pomeron alone (or a group of partners seen as a whole), the situation is more complicated when, for instance, the Pomeron and other contributors are considered simultaneously: the problem of the discrimination

of the intermediate states arises. To clarify this, let us illustrate the procedures using the standard or *ordinary* eikonal (OE) model [2] and the more sophisticated quasi-eikonal (QE) model [3], before we introduce our generalized eikonal (GE) model, in the case of the elastic pp and $\bar{p}p$ elastic scattering.

Consider the separate form of the Born amplitude

$$A_{pp,\text{Born}}^{\bar{p}p}(s, t) = a_+(s, t) \pm a_-(s, t), \quad (1)$$

where the crossing even part takes into account the Pomeron and the f Reggeon while the crossing odd part takes into account the Odderon and the ω Reggeon

$$\begin{aligned} a_+(s, t) &= a_P(s, t) + a_f(s, t), \\ a_-(s, t) &= a_O(s, t) + a_\omega(s, t). \end{aligned} \quad (2)$$

Let the corresponding crossing even and crossing odd input amplitudes in the impact parameter b -representation be $h_\pm(s, b)$, half the eikonal function $\chi_\pm(s, b)$

$$\begin{aligned} h_\pm &\equiv h_\pm(s, b) = \frac{1}{2} \chi_\pm(s, b) \\ &= \frac{1}{2s} \int_0^\infty dq q J_0(bq) a_\pm(s, -q^2). \end{aligned} \quad (3)$$

Knowledge of the eikonal amplitude in the b -representation, $H_{pp}^{\bar{p}p}(s, b)$, in terms of the Born components, $h_\pm(s, b)$, is at the basis of the eikonalization procedure since in all eikonal models, once $H_{pp}^{\bar{p}p}(s, b)$ is known, its inverse Fourier–Bessel transform leads finally to the eikonal amplitude in the (s, t) -space, $A_{pp,\text{Eik}}^{\bar{p}p}(s, t)$, to be used in

^a e-mail: desgrolard@ipnl.in2p3.fr

^b e-mail: giffon@ipnl.in2p3.fr

^c e-mail: martynov@bitp.kiev.ua

^d e-mail: predazzi@to.infn.it

¹ The critical Pomeron with $\delta_P = \delta_{\text{cr}}$ is a solution of the Pomeron equation within the Regge field theory [6, 7]; in this theory the supercritical Pomeron has $\delta_P > \delta_{\text{cr}}$.

the calculation of the observables:

$$A_{pp,\text{Eik}}^{\bar{p}p}(s, t) = 2s \int_0^\infty db b J_0(b\sqrt{-t}) H_{pp}^{\bar{p}p}(s, b). \quad (4)$$

The standard OE amplitude [2] in the impact parameter representation, $H_{pp,OE}^{\bar{p}p}(s, b)$, is obtained as a sum of all rescattering diagrams in the approximation for which there are only two nucleons on the mass shell in any intermediate state:

$$\begin{aligned} H_{pp,OE}^{\bar{p}p}(s, b) &= \frac{1}{2i} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2ih_+)^n (\pm 2ih_-)^m}{n!m!} - 1 \\ &= \frac{1}{2i} \{ \exp[2i(h_+ \pm h_-)] - 1 \}, \end{aligned} \quad (5)$$

where -1 subtracts the term with $n = m = 0$. This limitation neglects the possibility of taking in consideration intermediate multiparticle states. In a QE model [3] the effect of these multiparticle states is taken into account generalizing in a phenomenological way the various exchange diagrams. This is realized introducing an additional parameter λ ($\lambda = 1$ corresponds to OE). The eikonalized amplitude in the b -representation then becomes

$$\begin{aligned} H_{pp,QE}^{\bar{p}p}(s, b) &= \frac{1}{2i\lambda} \\ &\times \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (\lambda)^{n+m} \frac{(2ih_+)^n (\pm 2ih_-)^m}{n!m!} - 1 \right) \\ &= \frac{1}{2i\lambda} \{ \exp[2i\lambda(h_+ \pm h_-)] - 1 \}. \end{aligned} \quad (6)$$

However, it is not clear that all intermediate states between two Pomerons (or one Pomeron and one Odderon etc.) can be described by the same parameter λ . As soon as more Reggeons come into the game several different parameters could be used. There follows a much more involved formalism. From a phenomenological point of view, it is necessary to note here, as a motivation of generalizing the actual methods of eikonalization, that the QE method generally does not lead to a good agreement with pp and $\bar{p}p$ data [8].

The present paper is a generalization of the QE method: specifically, we go from a one-parameter formalism to a three-parameters formalism to eikonalize the pp and $\bar{p}p$ amplitudes. In Sect. 2, as a first step, we deal with the case where the three parameters entering in the eikonalization procedure obey a specific relation. In Sect. 3, we discuss the general three-parameter eikonalized $\bar{p}p$ and pp amplitudes where the intermediate states between Pomeron–Pomeron, Pomeron–Odderon and Odderon–Odderon exchanges are taken into account. General expressions will be given in simple, closed analytic form. The constraints arising from unitarity can then be studied in the cases where the number of parameters is two or three. Especially interesting in the latter case is that the Odderon intercept must be below unity.

2 Eikonalization of the elastic amplitude with two parameters describing the intermediate states

2.1 Amplitudes

In the following, we simplify the discussion by using two *Reggeons* only with a definite C -parity, which we call for brevity the Pomeron (P) and the Odderon (O). Actually the Pomeron together with the f Reggeon acts as a first *Reggeon* $\tilde{P} = (P + f)$ and the Odderon together with the ω Reggeon acts as a second *Reggeon* $\tilde{O} = (O + \omega)$. In what follows (Sects. 2 and 3) we consider for simplicity the case of $\bar{p}p$ scattering only (changes for pp are self-evident); later, we return to the general case by writing the final analytic amplitudes and discussing the unitarity constraints.

In the QE model [3], we do not discriminate as to the intermediate states between \tilde{P} – \tilde{P} , \tilde{O} – \tilde{O} and \tilde{P} – \tilde{O} . Releasing this assumption gives rise to a new generalized eikonal (GE) model in which the three intermediate states can be different. We proceed step by step. In the first step, each intermediate state between the vertices *particle–Reggeon–particle* is correlated with a quantity which we denote by $\lambda_i^{1/4} \lambda_k^{1/4}$ ($i, k = +, -$) where the positive and energy-independent parameters λ_i and λ_k depend on which *Reggeons* are exchanged to the left and to the right sides of a given diagram. The (somewhat curious) $1/4$ power is chosen so as to obtain the expressions $\lambda_\pm h_\pm$ for the final amplitudes.

In terms of these two parameters, we define the three coefficients

$$\begin{aligned} \lambda_+ &= C_{PP}, & \lambda_- &= C_{OO} \\ \text{and} & & \lambda_0 &= C_{PO} = C_{OP}, \end{aligned} \quad (7)$$

where the relation

$$\lambda_0^2 = \lambda_+ \lambda_- \quad (8)$$

has been assumed².

Such a procedure, roughly speaking, mimics a situation where the *particle–Pomeron–particle* and the *particle–Odderon–particle* amplitude vertices (g_+ and g_-) are rescaled by a priori different positive constants ($\lambda_+^{1/2}$ and $\lambda_-^{1/2}$). This means that, formally, we change the *coupling constants* of the *Reggeons* by

$$g_+ \rightarrow \sqrt{\lambda_+} g_+ \quad \text{for a } pPp\text{-vertex,}$$

$$g_- \rightarrow \sqrt{\lambda_-} g_- \quad \text{for a } pOp\text{-vertex.}$$

However, this implies that for each diagram, we have now extra multipliers originating from the vertices of the extreme left and right *Reggeons*. It is necessary to divide

² Strictly speaking, we should have used $C_{\tilde{P}\tilde{P}}$ etc. instead of C_{PP} etc. in (7) to recall that $\tilde{P} = P + f$ and $\tilde{O} = P + \omega$ but we decided not to do this; we will ignore this formal complication throughout the paper.

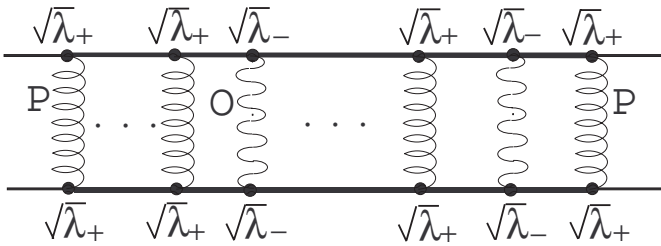


Fig. 1. A typical rescattering diagram with a Pomeron at both ends

the contributions of these diagrams by well-defined factors. There are, in fact, three kinds of rescattering diagrams with n Pomerons and m Odderons that one should consider (1) both the extreme left and the extreme right *Reggeons* are P (2) both the extreme left and the extreme right *Reggeons* are O (3) the extreme left (right) *Reggeon* is P while the extreme right (left) *Reggeon* is O (or vice versa). Consequently, the diagrams of the first type must be divided by λ_+ , the second ones by λ_- and the third ones by $(\lambda_- \lambda_+)^{1/2}$.

Consider now each kind of diagrams separately, beginning with the first kind (illustrated in the Fig. 1). We work out the case for $\bar{p}p$, but the same procedure applies to the pp case.

Within the present generalization, the contribution of all the diagrams (consisting in n Pomerons and m Odderons) is the following:

$$H[PP] = \frac{1}{2i\lambda_+} \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} (2i\lambda_+ h_+)^n (2i\lambda_- h_-)^m \times \frac{(n+m-2)!}{(n-2)!m!(n+m)!}. \tag{9}$$

The factor $1/(n+m)!$ takes into account the total number of $n+m$ *Reggeons*; the other factor $(n+m-2)!/(n-2)!m!$ (number of ways to choose m Odderons and $n-2$ Pomerons from the $n+m-2$ *Reggeons*) accounts for all permutations of non-identical Pomerons and Odderons keeping two Pomerons at the left and right ends of diagrams.

Similarly for the other contributions one obtains

$$H[OO] = \frac{1}{2i\lambda_-} \sum_{n=0}^{\infty} \sum_{m=2}^{\infty} (2i\lambda_+ h_+)^n (2i\lambda_- h_-)^m \times \frac{(n+m-2)!}{n!(m-2)!(n+m)!}, \tag{10}$$

$$H[PO] = H[OP] = \frac{1}{2i\sqrt{\lambda_+ \lambda_-}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (2i\lambda_+ h_+)^n (2i\lambda_- h_-)^m \times \frac{(n+m-2)!}{(n-1)!(m-1)!(n+m)!}. \tag{11}$$

Defining

$$H^{\bar{p}p}(s, b) = h_+ + h_- + H[PP] + H[OO] + 2H[PO], \tag{12}$$

the $\bar{p}p$ and the pp two-parameters eikonalized amplitudes in the GE case in the impact parameter representation take the final form (see Appendix A)

$$H^{\bar{p}p}_{pp,GE}(s, b) = h_+ \pm h_- + \left(\frac{h_+ \sqrt{\lambda_+} \pm h_- \sqrt{\lambda_-}}{h_+ \lambda_+ \pm h_- \lambda_-} \right)^2 \times \left(\frac{e^{2i(h_+ \lambda_+ \pm h_- \lambda_-)} - 1}{2i} - (h_+ \lambda_+ \pm h_- \lambda_-) \right). \tag{13}$$

Recall that, as stressed previously, this result is obtained in the case of two *Reggeons* irrespective of whether or not we consider them as being P and O only or whether we have grouped them together into the crossing even $\tilde{P} = P + f$ and the crossing odd $\tilde{O} = O + \omega$ combinations. Accordingly, we have the definitions (1–3), if the intermediate states “depend” only on the parity but not on the specific *Reggeon* (P, f or O, ω).

A similar compact formula can be obtained in the case of the so-called generalized U-matrix model [5]. With the same notation one obtains

$$H^{\bar{p}p}_{pp,GUM}(s, b) = \frac{(h_+ \pm h_-) \mp \frac{1}{2i} h_+ h_- (\sqrt{\lambda_+} - \sqrt{\lambda_-})^2}{1 - 2i(h_+ \lambda_+ \pm h_- \lambda_-)}. \tag{14}$$

Actually, here we confine ourselves to the simplest case when the input amplitudes are purely elastic. In the most general case, we should also introduce other amplitudes corresponding to different effective couplings at the extreme left and right ends of diagrams when the initial (or final) state in the corresponding vertex is not a single proton (similarly for the “internal amplitudes” inside the n -*Reggeon* diagrams). These new types of amplitude are the analog of those considered in diffraction dissociation (with not too high effective masses). By integration and summation over many intermediate states, new amplitudes would be derived by modifying appropriately each h_i and λ_i . The important difference would be that these new amplitudes would have different energy independent parts in their slopes (in agreement with the data) but, for large s , would reduce to the present amplitudes. We will not consider this additional complication here.

2.2 Unitarity constraints

The unitarity inequality

$$|H^{\bar{p}p}(s, b)| \leq 1 \tag{15}$$

restricts the admitted values for the parameters λ_+ and λ_- .

In Appendix B we briefly discuss the framework of the input (or *Born*) amplitudes to be used in the general scheme in both the (s, t) and (s, b) -representations, obtained the one from the other via a Fourier–Bessel transform. From the formulae of Appendix B, valid at high energy if the secondary *Reggeons* are neglected, one sees that the exponential term in (13) can be neglected because

h_+ becomes mainly imaginary and its modulus increases with energy. Thus, keeping the main and the next orders in h_-/h_+ , we obtain

$$\left| h_+ \pm h_- - h_+ \left(1 \pm 2\sqrt{\frac{\lambda_- h_-}{\lambda_+ h_+}} \mp \frac{\lambda_- h_-}{\lambda_+ h_+} + \frac{1}{2i} \frac{1}{h_+ \lambda_+} \right) \right| \leq 1,$$

which is the same as

$$\left| \frac{1}{2i\lambda_+} \mp h_- \left(1 - \frac{\sqrt{\lambda_-}}{\sqrt{\lambda_+}} \right)^2 \right| \leq 1. \tag{16}$$

If the crossing even (or Pomeron) and the crossing odd (or Odderon) trajectories are written as

$$\alpha_{\pm}(t) = 1 + \delta_{\pm} + \alpha'_{\pm} t, \tag{17}$$

the inequality (16) can be satisfied either if³

$$\lambda_+ = \lambda_-, \quad \text{and } \lambda_+ \geq 1/2, \quad \delta_- \geq 0, \tag{18}$$

or, when $\lambda_+ \neq \lambda_-$, if⁴

$$\lambda_- \text{ is arbitrary, } \lambda_+ \geq 1/2, \quad \delta_- \leq 0, \tag{19}$$

Similar results can be obtained for the two-parameters Generalized U -Matrix model (GUM) [5] in which case from (15) unitarity requires

$$\left| \frac{(h_+ \pm h_-) \mp \frac{1}{2i} h_+ h_- (\sqrt{\lambda_+} - \sqrt{\lambda_-})^2}{1 - 2i(h_+ \lambda_+ \pm h_- \lambda_-)} \right| \leq 1. \tag{20}$$

One sees at once that the second term of the numerator is dangerous for the unitarity restriction (15) because it dominates at $s \rightarrow \infty$ if $\delta_+ > 0$ and $\delta_- > 0$ and grows faster than the denominator. In this case, there are three solutions to prevent violation of unitarity

- (i) $\lambda_+ = \lambda_-$,
- (ii) $\delta_- \leq 0$,
- (iii) $\delta_+ \leq 0$.

Notice that, compared to the GE, in the GUM we have a third solution (iii) which is technically possible when $\delta_- > \delta_+$ in spite of its apparent non-realistic aspect [5]. To see how the above solutions arise we note that when $\delta_+ > \delta_-$ and $|h_+| \rightarrow \infty$, the only way to satisfy (20) is when $\lambda_+ = \lambda_-$ and $\lambda_+ \geq 1/2$ (the other two solutions arise similarly, see [5]).

Summarizing, we find that the two-parameters GE model fulfils the unitarity requirement (15) when

(i) either it reduces to the QE (U -matrix) model with one parameter ($\lambda = \lambda_+ = \lambda_-$), with

$$\lambda \geq 1/2, \quad \delta_{\pm} > 0, \tag{21}$$

³ When $s \rightarrow \infty$ and b is inside the interaction radius ($0 \leq b^2 < 4\alpha'_+ \delta_+ \ln^2 s = R^2$) [10, 11], $|h_-| \rightarrow \infty$ (see Appendix B).

⁴ No restriction is found on λ_- , because, in this case, $|h_-| \rightarrow 0$ when $s \rightarrow \infty$ and $0 \leq b^2 < R^2$ (see Appendix B).

Table 1. Summary of unitarity constraints of the GE and GUM procedures with two λ parameters. Besides the limitations on the λ_{\pm} parameters, we show those on the trajectories in the different cases together with the asymptotic cross section when the Pomeron (Odderon) dominates

	(i)	(ii)	(iii)
	U -matrix only		
λ_+	$= \lambda > 1/2$	$> 1/2$	
λ_-	$= \lambda > 1/2$		$> 1/2$
δ_+	$\geq \delta_-$	> 0	≤ 0
δ_-	> 0	≤ 0	> 0
α'_+	$\geq \alpha'_-$	$\geq \alpha'_-$	$\leq \alpha'_-$
$\sigma_{\text{tot}}^{pp, p\bar{p}} \propto$	$\alpha'_+ \delta_+ \ell n^2 s$	$\alpha'_+ \delta_+ \ell n^2 s$	$\alpha'_- \delta_- \ell n^2 s$

(ii) or it satisfies the limitations

$$\lambda_+ \geq 1/2, \quad \text{if } \delta_- \leq 0, \quad \delta_+ > 0, \tag{22}$$

(iii) or, finally, it obeys

$$\lambda_- \geq 1/2, \quad \text{if } \delta_+ \leq 0, \quad \delta_- > 0 \tag{23}$$

(only for the U -matrix).

These conditions have to be combined with the well-known Pomeron/Odderon hierarchy [10, 11]

$$\delta_+ \geq \delta_-, \quad \alpha'_+ \geq \alpha'_-,$$

for the case (21) and

$$\delta_- \geq \delta_+, \quad \alpha'_- \geq \alpha'_+,$$

for the U -matrix case (23) where α'_{\pm} are the slopes of the linear trajectories for the Pomeron (Odderon). These conclusions are shown in Table,1, where in order to emphasize the differences between the various classes of models, we give also the asymptotic behaviour of $\sigma_{\text{tot}}^{pp, p\bar{p}}$ (see [5]).

3 Eikonalization of the elastic amplitude with three parameters

3.1 Contributions to the amplitudes

We consider now the more general case where (8) is not valid,

$$\lambda_0^2 \neq \lambda_+ \lambda_-.$$

In this case, three different coefficients $\lambda_0, \lambda_+, \lambda_-$ should be considered. Again, as in Sect. 2, (let us remind P,(O) means here in fact $\tilde{P}, (\tilde{O})$) we deal with three types of diagrams and corresponding terms of the amplitudes $H[PP]$, $H[OO]$ and $H[PO]$. We start with the first term $H[PP]$

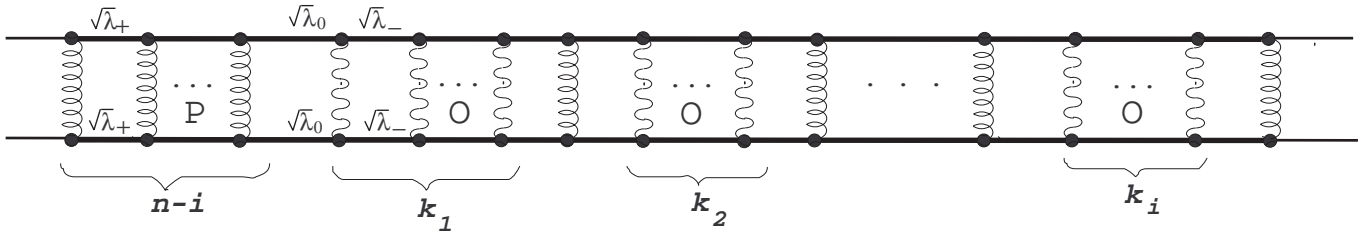


Fig. 2. A typical diagram with a Pomeron at both ends in the case when the Odderons are grouped in i cells

(see Fig. 2). There are n Pomerons and m Odderons, distributed in i cells. The maximal value of i is

$$\begin{matrix} n-1 & \text{if} & m > n-1 \\ m & \text{if} & m \leq n-1. \end{matrix}$$

Suppose that one Odderon is in each of the i cells. Then, the number of ways to choose i cells from the $n-1$ cells is

$$\binom{n-1}{i} = \frac{(n-1)!}{i!(n-1-i)!}.$$

The number of ways to distribute the remaining $m-i$ Odderons into the already chosen i cells is

$$\binom{(m-i)+i-1}{m-i} = \binom{m-1}{i-1} = \frac{(m-1)!}{(i-1)!(m-i)!}.$$

Therefore, the total number of diagrams with n Pomerons and m Odderons (distributed in i cells among the Pomerons) is

$$\binom{n-1}{i} \binom{m-1}{i-1}.$$

For each diagram, as exemplified in Fig. 2, we have a certain overall factor made of powers of λ_+ , λ_- and λ_0 which we now proceed to calculate. If there are k_1, k_2, \dots, k_i Odderons in the 1st, 2nd, \dots , i th cell, the ℓ th cell contributes with the factor $\lambda_-^{k_\ell-1}$. Consequently all the cells give the factor

$$\lambda_-^{k_1-1} \lambda_-^{k_2-1} \dots \lambda_-^{k_i-1} = \lambda_-^{k_1+k_2+\dots+k_i-i} = \lambda_-^{m-i},$$

because $k_1+k_2+\dots+k_i = m$. In addition, each cell gives λ_0^2 ; therefore for all the cells we have $(\lambda_0^2)^i$. Furthermore, the number of cells without Odderons inserted is $n-1-i$ and this gives a factor λ_+^{n-1-i} . Thus, the total factor for this diagram is

$$(\lambda_0^2)^i \lambda_-^{m-i} \lambda_+^{n-1-i} = \frac{1}{\lambda_+} \left(\frac{\lambda_0^2}{\lambda_+ \lambda_-} \right)^i \lambda_+^n \lambda_-^m.$$

Besides this, the factor $2ih_+$ corresponds to each Pomeron amplitude and the factor $2ih_-$ corresponds to each Odderon amplitude; and remember there are altogether n Pomeron and m Odderon amplitudes.

Summing everything up, the contribution to the re-scattering amplitude of all diagrams with Pomerons at

the left and right ends of each diagram has the following form:

$$\begin{aligned} 2i\lambda_+ H[PP] &= \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \sum_{i=1}^m \frac{1}{(m+n)!} \binom{n-1}{i} \binom{m-1}{i-1} \\ &\times \left(\frac{\lambda_0^2}{\lambda_+ \lambda_-} \right)^i (2ih_+ \lambda_+)^n (2ih_- \lambda_-)^m \\ &+ \sum_{n=2}^{\infty} \sum_{m=n}^{\infty} \sum_{i=1}^{n-1} \frac{1}{(m+n)!} \binom{n-1}{i} \binom{m-1}{i-1} \\ &\times \left(\frac{\lambda_0^2}{\lambda_+ \lambda_-} \right)^i (2ih_+ \lambda_+)^n (2ih_- \lambda_-)^m \\ &+ \sum_{n=2}^{\infty} \frac{1}{n!} (2ih_+ \lambda_+)^n, \end{aligned} \tag{24}$$

the last term taking into account the diagrams without Odderons.

The same form but with the replacements $h_+ \leftrightarrow h_-$ and $\lambda_+ \leftrightarrow \lambda_-$ holds for $H[OO]$. Similarly, one obtains for the contribution of diagrams with a Pomeron at one end and an Odderon at the other one

$$\begin{aligned} 2i\lambda_0 H[PO] &= \sum_{n=1}^{\infty} \sum_{m=1}^n \sum_{i=1}^m \frac{1}{(m+n)!} \binom{n-1}{i-1} \binom{m-1}{i-1} \left(\frac{\lambda_0^2}{\lambda_+ \lambda_-} \right)^i \\ &\times (2ih_+ \lambda_+)^n (2ih_- \lambda_-)^m \\ &+ \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \sum_{i=1}^n \frac{1}{(m+n)!} \binom{n-1}{i-1} \binom{m-1}{i-1} \left(\frac{\lambda_0^2}{\lambda_+ \lambda_-} \right)^i \\ &\times (2ih_+ \lambda_+)^n (2ih_- \lambda_-)^m. \end{aligned} \tag{25}$$

3.2 Analytical form of the amplitude

3.2.1 General considerations

It is somewhat surprising that one can obtain a compact analytical form of the total amplitude. To do this, we begin by summing over i in the previous expressions. Introducing

$$z = \frac{\lambda_0^2}{\lambda_+ \lambda_-}, \tag{26}$$

and setting $N = m$ (or $N = n-1$) in (24), we obtain

$$\sum_{i=1}^N \binom{n-1}{i} \binom{m-1}{i-1} z^i$$

$$\begin{aligned}
 &= \frac{z}{m} \frac{d}{dz} \sum_{p=1}^m \frac{z^p}{(p!)^2} (1-n)_p (-m)_p \\
 &= \frac{z}{m} \frac{d}{dz} {}_2F_1(-m, 1-n; 1; z) \\
 &= z(n-1) {}_2F_1(1-m, 2-n; 2; z),
 \end{aligned}$$

where $(a)_b = \Gamma(a+b)/\Gamma(a)$ is the Pochhammer symbol and the definition of the hypergeometric function ${}_2F_1$ has been used in [12]. Similarly, setting $N = m$ (or $N = n$) in (25), we get

$$\sum_{i=1}^N \binom{n-1}{i-1} \binom{m-1}{i-1} z^i = z {}_2F_1(1-m, 1-n; 1; z).$$

Substituting these results we get

$$\begin{aligned}
 &2i\lambda_+ H[PP] \\
 &= z \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \frac{x^n y^m}{(n+m)!} (n-1) {}_2F_1(1-m, 2-n; 2; z) \\
 &+ e^x - x - 1,
 \end{aligned} \tag{27}$$

and

$$2i\lambda_0 H[PO] = z \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{x^n y^m}{(n+m)!} {}_2F_1(1-m, 1-n; 1; z), \tag{28}$$

where

$$x = 2i\lambda_+ h_+, \quad y = 2i\lambda_- h_-. \tag{29}$$

We obtain $\lambda_- H[OO]$ from $\lambda_+ H[PP]$ with the replacement $x \longleftrightarrow y$.

To perform the remaining summations, we use the following integral representation which defines ${}_2F_1(a, b; c; z)$ as an analytic function in the z -plane, by means of the contour integral [13]

$$\begin{aligned}
 {}_2F_1(a, b, c; z) &= \frac{-i\Gamma(c) \exp(-i\pi b)}{2\Gamma(b)\Gamma(c-b) \sin \pi b} \\
 &\times \oint_C dt t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a},
 \end{aligned} \tag{30}$$

where

$$\Re c > \Re b, \quad |\arg(-z)| < \pi, \quad b \neq 1, 2, 3, \dots,$$

and where the integration contour is defined in Fig. 3.

3.2.2 Determination of $H[PO]$

Consider first $H[PO]$. Equations (28) and (30) yield

$$\begin{aligned}
 2i\lambda_0 H[PO] &= \frac{z}{2\pi i} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{x^n y^m}{\Gamma(n+m+1)} \\
 &\times \oint_C dt t^{-n} (t-1)^{n-1} (1-tz)^{m-1}
 \end{aligned}$$

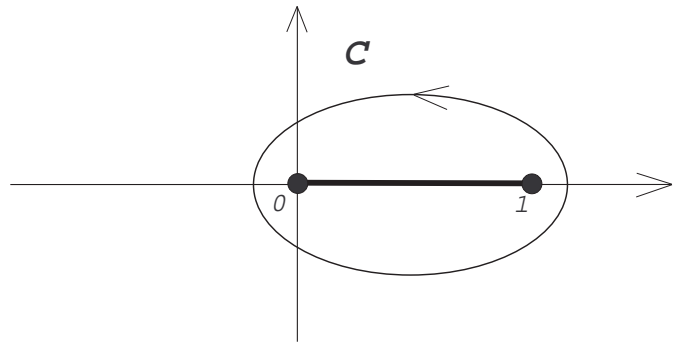


Fig. 3. Integration contour in the complex t -plane

$$\begin{aligned}
 &= \frac{z}{2\pi i} \oint_C \frac{dt}{(t-1)(1-tz)} \\
 &\times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{X^n Y^m}{\Gamma(n+m+1)},
 \end{aligned} \tag{31}$$

where we have introduced the t -dependent variables

$$X = x(t-1)/t, \quad Y = y(1-tz). \tag{32}$$

The sum over n and m is easily calculated (see Appendix C) and one obtains

$$\begin{aligned}
 S(X, Y) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{X^n Y^m}{\Gamma(n+m+1)} \\
 &= 1 + \frac{X}{Y-X} e^Y - \frac{Y}{Y-X} e^X.
 \end{aligned} \tag{33}$$

Thus (31) splits into three integrals:

$$\begin{aligned}
 2i\lambda_0 H[PO] &= \frac{z}{2\pi i} \oint_C \frac{dt}{(t-1)(1-tz)} \\
 &\times \left(1 + \frac{X}{Y-X} e^Y - \frac{Y}{Y-X} e^X \right) \\
 &\equiv I_1 + I_2 + I_3.
 \end{aligned} \tag{34}$$

These integrals are easily calculated. There is only one pole at $t = 1$ for the first integral, and no pole inside the contour for the second one. Thus,

$$I_1 = \frac{z}{1-z}, \quad I_2 = 0. \tag{35}$$

The integrand in the third integral

$$I_3 = -\frac{z}{2\pi i} \oint_C \frac{dt}{(t-1)} \cdot \frac{t y e^{x(t-1)/t}}{t y (1-tz) + x(1-t)}$$

has poles at $t = 1$ and $t = t_{\pm}$, where

$$t_{\pm} = \frac{1}{2yz} (-x + y \pm \sqrt{(x-y)^2 + 4xyz}).$$

As in the previous case only the pole at $t = 1$ is inside the integration contour. However, in this case the

integrand has an essential singularity at $t = 0$ due to $\exp(x(t - 1/t))$. One can, however, sum the residues over all the poles outside the contour C (this can be done because the integrand vanishes at $|t| \rightarrow \infty$ faster than $1/|t|$). Thus I_3 is obtained from the sum of the residues at $t = t_-$ and $t = t_+$, taken with opposite signs:

$$I_3 = -\frac{1}{(t_+ - t_-)} \left(\frac{e^{u_+}}{1 - 1/t_+} - \frac{e^{u_-}}{1 - 1/t_-} \right), \tag{36}$$

defining

$$u_{\pm} = \frac{1}{2}(x + y \pm \sqrt{(x - y)^2 + 4xyz}).$$

Collecting (35) and (36), we finally get

$$2i\lambda_0 H[PO] = \frac{z}{1 - z} + \frac{zxy}{\sqrt{(x - y)^2 + 4xyz}} \left(\frac{e^{u_+}}{u_+} - \frac{e^{u_-}}{u_-} \right). \tag{37}$$

3.2.3 Determination of $H[PP]$

Consider now the case of $H[PP]$. The contribution of all diagrams with the Pomerons to the left and right ends of each diagram has the form (27); we have

$$\begin{aligned} 2i\lambda_+ H[PP] &= z \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \frac{x^n y^m}{(n + m)!} \frac{1}{m} \frac{d}{dz} \\ &\times {}_2F_1(-m, 1 - n; 1; z) + e^x - x - 1 \\ &\equiv e^x - x - 1 + I_{PP}. \end{aligned} \tag{38}$$

Repeating the arguments given for $H[PO]$, we can rewrite I_{PP} as

$$\begin{aligned} I_{PP} &= \frac{z}{2\pi i} \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \frac{x^n y^m}{(n + m)!} \frac{1}{m} \frac{d}{dz} \\ &\times \oint_C dt t^{-n} (t - 1)^{n-1} (1 - tz)^m \\ &= -\frac{z}{2\pi i} \oint_C dt \frac{t}{(t - 1)(1 - tz)} \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \frac{X^n Y^m}{(n + m)!} \end{aligned} \tag{39}$$

with the same X, Y (see (32)) and the same contour C as above (see Fig. 3). The sum over n and m can now immediately be obtained. One finds

$$\sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \frac{X^n Y^m}{(n + m)!} = S(X, Y) - \tilde{S}(X, Y),$$

where $S(X, Y)$ was given in (33) and

$$\tilde{S}(X, Y) = \frac{X}{Y} (e^Y - Y - 1). \tag{40}$$

The integral (39) can thus be rewritten in the following form:

$$\begin{aligned} I_{PP} &= -\frac{z}{2\pi i} \oint_C dt \frac{t}{(t - 1)(1 - tz)} \\ &\times [S(X, Y) - \tilde{S}(X, Y)] \equiv \mathcal{I}_{PP} - \tilde{\mathcal{I}}_{PP}. \end{aligned} \tag{41}$$

Replacing X and Y by their expressions (32), $\tilde{\mathcal{I}}_{PP}$ becomes

$$\tilde{\mathcal{I}}_{PP} = \frac{z}{2\pi i} \oint_C dt \frac{x/y}{(1 - tz)^2} (e^{y(1-tz)} - y(1 - tz) - 1), \tag{42}$$

which is zero, because there are no singularities inside the integration contour. Now we examine \mathcal{I}_{PP} , the first integral in (41); the only difference with the previous $[PO]$ case (34) is the factor t in the integrand and we may write

$$\begin{aligned} \mathcal{I}_{PP} &= -\frac{z}{2\pi i} \oint_C dt \frac{t}{(t - 1)(1 - tz)} \\ &\times \left[1 + \frac{X}{Y - X} e^Y - \frac{Y}{Y - X} e^X \right] \equiv \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned} \tag{43}$$

We see immediately that the first and the second terms are

$$\mathcal{I}_1 = -\frac{z}{1 - z}, \quad \mathcal{I}_2 = 0, \tag{44}$$

while the third term can be rewritten as

$$\mathcal{I}_3 = \frac{zy}{2\pi i} \oint_C dt \frac{t^2 e^{x(t-1)/t}}{(t - 1)[ty(1 - tz) - x(t - 1)]}. \tag{45}$$

We always have an essential singular point at $t = 0$ because of the exponential. At the same time there are three poles outside the contour, namely, at $t = t_-, t = t_+$ (defined in the previous subsection for $[PO]$) and at infinity. Thus, we can expand the contour C at very large distances from $t = 0$, take the residues at $t = t_-$ and at $t = t_+$ (with minus sign) and write

$$\mathcal{I}_3 = -zy \text{Res}_{t=t_-} f(t) - zy \text{Res}_{t=t_+} f(t) + \frac{zy}{2\pi i} \oint_R dt f_{as}(t).$$

Here R is a circle of large radius and $f(t)$ is the integrand in (45):

$$f(t) = \frac{t^2 e^{x(t-1)/t}}{(t - 1)[ty(1 - tz) - x(t - 1)]}.$$

On the circle of large radius we can approximate $f(t)$ by its asymptotic form for $|t| \rightarrow \infty$, i.e. $f_{as}(t) = -e^x/tyz$. Thus

$$\frac{zy}{2\pi i} \oint_R dt f_{as}(t) = -\frac{e^x}{2\pi i} \oint_R \frac{dt}{t} = -e^x.$$

Collecting this integral and the residues at $t = t_-, t = t_+$ we obtain for \mathcal{I}_3 the following expression:⁵

$$\mathcal{I}_3 = -e^x + \frac{t_-^2 e^{x-x/t_-}}{(t_- - 1)(t_- - t_+)} + \frac{t_+^2 e^{x-x/t_+}}{(t_+ - 1)(t_+ - t_-)}. \tag{46}$$

⁵ It is possible, of course, to obtain this result more exactly replacing the integration variable in (45) by $1/t$. After this transformation the singularities of interest are inside a new integration contour.

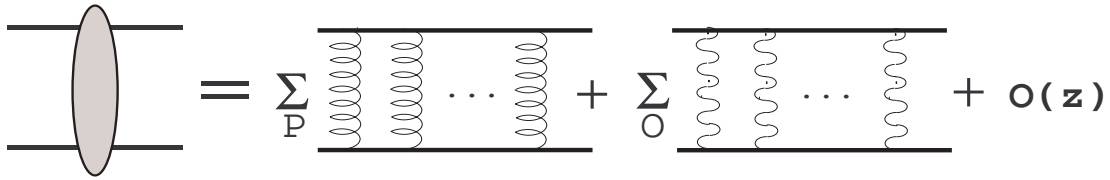


Fig. 4. Diagram representation of $H(s, b)$ at $z \ll 1$

Taking into account (38) and the expressions for t_{\pm} we obtain the final expression for $H[PP]$

$$2i\lambda_+H[PP] = -x - 1 - \frac{z}{1-z} - \frac{x}{2\sqrt{(x-y)^2 + 4xyz}}$$

$$\times \left\{ (-x + y - \sqrt{(x-y)^2 + 4xyz}) \frac{e^{u_+}}{u_+} - (-x + y + \sqrt{(x-y)^2 + 4xyz}) \frac{e^{u_-}}{u_-} \right\} \quad (47)$$

3.2.4 Full eikonalized amplitude

The full eikonalized amplitude (12), is now obtained by adding the two rescattering contributions found above to the Born amplitude and using the substitution $x \longleftrightarrow y$ to get $\lambda_-H[OO]$ from $\lambda_+H[PP]$. Putting everything together, we finally have the three-parameter eikonalized amplitudes for the $\bar{p}p$ and pp elastic scattering process:

$$H_{pp,GE}^{\bar{p}p}(s, b) = \frac{i}{2(\lambda_0^2 - \lambda_+\lambda_-)} \left\{ a + e^{i(\lambda_+h_+ \pm \lambda_-h_-)} \right. \quad (48)$$

$$\left. \times \left[-a \cos \phi_{\pm} + i \frac{c_+h_+ \pm c_-h_-}{\phi_{\pm}} \sin \phi_{\pm} \right] \right\},$$

where we have introduced the three constants a and c_{\pm} defined as

$$a = 2\lambda_0 - \lambda_+ - \lambda_-, \quad (49)$$

$$c_{\pm} = \lambda_+\lambda_- - 2\lambda_0^2 - \lambda_{\pm}^2 + 2\lambda_0\lambda_{\pm}, \quad (50)$$

in terms of the parameters of the model and the functions (of s and b)

$$\phi_{\pm} = \sqrt{(\lambda_+h_+ \mp \lambda_-h_-)^2 \pm 4\lambda_0^2h_+h_-}. \quad (51)$$

3.3 Particular cases

After finding general expressions for the amplitudes, valid at any z , we investigate some particular cases.

(i) $z = 1$ or $\lambda_0^2 = \lambda_+\lambda_-$.

We may use the results of Sect. 3.1 to check the results of Sect. 2, relative to the two-parameter parameterization. In fact (9)–(13) directly follow from (27) and (28) if ${}_2F_1$ is calculated at $z = 1$ using (30).

(ii) $z = 1 - \varepsilon$, $|\varepsilon| \ll 1$.

It is instructive to derive an expansion of the general expressions (37) and (47) in terms of the small quantity

$\varepsilon = 1 - z$. Consider for example (37) for $H[PO]$. One can see that the first term $z/(1-z)$ and the third one with $\exp(u_-)/u_-$ have singularities at $\varepsilon = 0$ which cancel each other. Keeping the zeroth and first powers of ε we obtain the following result for $H[PO]$ at $z \approx 1$

$$2i\lambda_0H[PO] \simeq \frac{xy}{(x+y)^2} \left\{ e^{x+y} - x - y - 1 - \varepsilon[-x - y - 1 + \frac{xy}{2(x+y)^2}((x+y)^2 + 4(x+y) + 6) + e^{x+y} \left[1 + \frac{xy}{x+y} - \frac{3xy}{(x+y)^2} \right]] \right\}. \quad (52)$$

Similarly the singularities at $\varepsilon = 0$ cancel in $H[PP]$ and we obtain

$$2i\lambda_+H[PP] \simeq \frac{x^2}{(x+y)^2} \left\{ e^{x+y} - x - y - 1 + \varepsilon \frac{y}{(x+y)^2} \left[\frac{1}{2}(x+y)(y^2 + xy + 2y - 2x - 4) + 3y + e^{x+y}(2x - y - x^2 - xy) \right] \right\}. \quad (53)$$

Using (12) we obtain the full eikonalized amplitude in the b -representation. Of course at $\varepsilon = 0$ (or $z = 1$) these results coincide with the similar expressions of the Sect. 2.

(iii) $z \ll 1$ or $\lambda_0^2 \ll \lambda_+\lambda_-$ (weak coupling limit).

This case is of conceptual interest because it corresponds to a clear separation between the Pomeron contribution and the Odderon one. As a result of this approximation, we obtain eikonalized Pomeron, eikonalized Odderon and small interference terms proportional, respectively, to z/λ_0 , z/λ_+ , z/λ_- . This is illustrated in Fig. 4. The expressions for $H_{pp}^{\bar{p}p}$ can easily be obtained.

(iv) $z \gg 1$ (strong coupling limit).

This case is the more complicated one to obtain approximate expressions for $H_{pp}^{\bar{p}p}$ in closed analytical form and we will not give here the formulae that are in order. Preliminary numerical investigations, however, seem to indicate that this is the choice that gives the best account of the data.

3.4 Unitarity constraints

We restrict our considerations to the case

$$|h_-(s, b)| \ll |h_+(s, b)|. \quad (54)$$

which is the correct assumption in the high energy region (at least for $0 \leq b^2 < R^2$) when the Pomeron dominates over the Odderon and other Reggeons (see Appendix B). If we expand the full eikonalized amplitude (48) in terms of the small complex quantity $\epsilon = h_-(s, b)/h_+(s, b)$, we have

$$\phi_{\pm} \approx \lambda_+ h_+ \left(1 \mp \epsilon \left(\frac{\lambda_-}{\lambda_+} - 2 \frac{\lambda_0^2}{\lambda_+^2} \right) \right), \quad (55)$$

$$\frac{c_+ h_+ \pm c_- h_-}{\phi_{\pm}} \approx \frac{c_+}{\lambda_+} \left[1 \pm \epsilon \left(\frac{c_-}{c_+} + \frac{\lambda_-}{\lambda_+} - 2 \frac{\lambda_0^2}{\lambda_+^2} \right) \right], \quad (56)$$

$$H_{pp}^{\bar{p}p} \approx \frac{i}{2(\lambda_0^2 - \lambda_+ \lambda_-)} \left\{ a + \frac{1}{2} \left(-a + \frac{c_+}{\lambda_+} \right) e^{2i(\lambda_+ h_+ \pm h_- (\lambda_0^2 / \lambda_+))} - \frac{1}{2} \left(a + \frac{c_+}{\lambda_+} \right) e^{\pm 2i\lambda_- h_- (1 - (\lambda_0^2 / (\lambda_+ \lambda_-)))} \right\}. \quad (57)$$

The second term in the last expression can be neglected because $h_+(s, b)$ is mainly imaginary and $|h_+| \rightarrow \infty$ when $s \rightarrow \infty$ if $0 \leq b^2 < R^2$ (see Appendix B). We obtain

$$H_{pp}^{\bar{p}p} \approx \frac{i}{2(\lambda_0^2 - \lambda_+ \lambda_-)} \left\{ a - \frac{1}{2} \left(a + \frac{c_+}{\lambda_+} \right) e^{\pm 2i\lambda_- h_- (1-z)} \right\}. \quad (58)$$

The following comments apply.

(1) If $\delta_- > 0$, the unitarity inequality

$$|H_{pp}^{\bar{p}p}(s, b)| \leq 1 \quad (59)$$

cannot be satisfied unless the factor in front of the exponential in (58) is zero, since $|\exp(\pm 2i\lambda_- h_- (1-z))| \rightarrow \infty$ at $s \rightarrow \infty$ either for pp or for $\bar{p}p$. Thus, in general, the only possibility to preserve the unitarity restriction is $\delta_- \leq 0$. In this case $|h_-| \rightarrow 0$ when $s \rightarrow \infty$ and $0 \leq b^2 < R^2$ (see Appendix B) and consequently

$$H_{pp}^{\bar{p}p}(s, b) \approx \frac{i}{2} \left\{ \frac{2\lambda_0 - \lambda_+ - \lambda_-}{\lambda_+ \lambda_- (z-1)} + \frac{(\lambda_+ - \lambda_0)^2}{\lambda_+^2 \lambda_- (z-1)} \right\} = \frac{i}{2\lambda_+}. \quad (60)$$

As a consequence, if $\delta_- \leq 0$ the restriction

$$\lambda_+ \geq 1/2 \quad (61)$$

follows from the unitarity constraint. Let us emphasize that there are no restrictions on λ_- and λ_0 so long as $z = \lambda_0^2 / \lambda_- \lambda_+ \neq 1$.

(2) If

$$\lambda_0 = \lambda_+, \quad (62)$$

however, the factor in front of the exponential in (58) vanishes and the amplitude can be approximated as

$$H_{pp}^{\bar{p}p}(s, b) \approx \frac{i}{2} \left\{ \frac{2\lambda_0 - \lambda_+ - \lambda_-}{\lambda_+ \lambda_- (z-1)} \right\} = \frac{i}{2\lambda_+} \quad (63)$$

and we find again that (61) is valid irrespective of the sign of δ_- . Finally, if $\alpha_+(0) > \alpha_-(0)$ (as always assumed to be the case), the condition (61) can be satisfied if

$$\lambda_0 = \lambda_+ \neq \lambda_-. \quad (64)$$

Table 2. Summary of the results of Sects. 2.2, 3.4 for four classes of eikonalized models dominated by the Pomeron, respecting unitarity constraints

	$z = 1$ (QE)	$z = 1$ (GE)	$z \neq 1$ (GE)	$z \neq 1$ (GE)
λ_+	$\geq 1/2$	$\geq 1/2$	$\geq 1/2$	$\geq 1/2$
λ_-	$= \lambda_+$	any	any	any
λ_0	$= \lambda_+$	$(\lambda_+ \lambda_-)^{1/2}$	$= \lambda_+$	any
δ_+	$\geq \delta_-$	> 0	$\geq \delta_-$	> 0
δ_-	> 0	≤ 0	> 0 or ≤ 0	≤ 0

Table 2 collects the results found here and in Sect. 2.2 concerning the unitarization constraints on the parameters assuming Pomeron dominance, i.e. $\sigma_{\text{tot}}^{pp, \bar{p}p} \propto \alpha'_+ \delta_+ \ln^2 s$.

4 Conclusion

First of all, let us remind the reader that we have considered only an eikonalization procedure rather than a complete unitarization and that we have analyzed only the case of elastic scattering. As mentioned at the end of Sect. 2.1, the case of diffraction dissociation would imply the insertion of new couplings. With these new effective couplings, the main difference of the diffraction dissociation amplitudes derived in analogy with the elastic ones would be in the energy independent parts of their slopes (which, in fact, is experimentally quite different). Asymptotically, these more general amplitudes have similar behaviors and the results from unitarity would remain the same.

A recent controversy concerns the sign of $\delta_- = \delta_O$, (i.e. of the difference with unity of the Odderon intercept at $t = 0$). This quantity, (for which indications have been found long ago [9] that it should be negative) was initially [16] believed to be positive from QCD calculations of the Odderon trajectory, but counterarguments were then given [17] that δ_- should actually be negative. More recently [18], this parameters has been calculated and found to be negative indeed⁶. On purely phenomenological grounds, but with absolute rigor, we can state, in the more general case of Sect. 3, that δ_- *must* be negative or null unless the specific equality $\lambda_+ = \lambda_0$ holds in which case this sign can also be positive. Of course, nothing prevents a priori such an equality between quantities related to the coupling of particles to the Pomeron and the Odderon to hold, but it certainly looks like a rather peculiar relation and should it turn out to be indeed valid, it certainly would deserve further investigation to understand its implications.

The main merit of our paper, however, lies in the great generality of the formalism we have developed leading to a complete three-parameter eikonalization. In spite of the

⁶ The latest QCD calculation [19] gives $\delta_O = 0$.

apparent complications introduced, it is quite likely that these are not of a purely abstract interest and, indeed, we are already exploring its phenomenological implications in describing all elastic pp and $\bar{p}p$ data simultaneously [8]. The standard eikonal, in fact, is not suitable for a realistic physical description of the elastic amplitude where two classes of Reggeons contributions have to be kept: e.g. the Pomeron (essential to describe the small $|t|$ domain) and the Odderon (essential to account for the large $|t|$ domain). This is especially visible when discussing the $\bar{p}p$ and the pp scattering at high energy. In fact, in this case, as we have seen, it may be appropriate to group together the crossing even and the crossing odd combinations $P+f$ and $O+\omega$ but one could also argue that it is not necessary to eikonalize the secondary Reggeons (because they do not imply any violation of unitarity) in which case one would interpret the crossing even and the crossing odd parts as due simply to the Pomeron and the Odderon.

We believe that the present generalization can be successfully applied to a phenomenological description of all high energy pp and $\bar{p}p$ elastic scattering data where both the Pomeron and the Odderon contribute. This is under investigation presently [8].

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Appendix A Two-parameter amplitude (Proof of (13))

We consider for example (9) and rewrite

$$H[PP] = \frac{1}{2i\lambda_+} \phi_{PP}(x, y),$$

defining $x = 2i\lambda_+ h_+$, $y = 2i\lambda_- h_-$ and

$$\phi_{PP}(x, y) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^{k+2} y^m}{k! m! (k+m+2)(k+m+1)}.$$

After the substitution $y = xu$ this equation becomes

$$\begin{aligned} \phi_{PP}(x, y) &= \tilde{\phi}_{PP}(x, u) \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^{k+m+2} u^m}{k! m! (k+m+2)(k+m+1)} \end{aligned}$$

which satisfies

$$\tilde{\phi}_{PP}(x, 0) = \frac{\partial \tilde{\phi}_{PP}}{\partial x}(x, 0) = 0$$

and

$$\frac{\partial^2 \tilde{\phi}_{PP}}{\partial x^2}(x, u) = e^{x+xu}.$$

Its integration leads to

$$\phi_{PP}(x, y) = x^2 \varphi(x, y)$$

with

$$\varphi(x, y) = \frac{(e^{x+y} - 1)}{(x+y)^2} - \frac{1}{x+y}.$$

Finally, we obtain

$$H[PP] = \frac{x^2}{2i\lambda_+} \varphi(x, y).$$

Similarly

$$\begin{aligned} H[OO] &= \frac{y^2}{2i\lambda_-} \varphi(x, y), \\ H[PO] &= \frac{xy}{2i\sqrt{\lambda_+ \lambda_-}} \varphi(x, y). \end{aligned}$$

Collecting these results, we derive the final expression (13) for the impact parameter eikonalized amplitude in the two-parameters GE procedure.

Appendix B Born Pomeron and Odderon amplitudes

In this Appendix we give the expressions of the amplitudes which we use to discuss the unitarity constraints. For simplicity, we consider the specific model known as the Regge monopole (see below) but, as we will argue, the conclusions are quite model independent. For the same reason (simplicity), the input crossing even and odd (*Born*) amplitudes entering in (1) are approximated, at high energy, by the Pomeron and Odderon contributions, respectively:

$$a_+(s, t) = a_P(s, t), \quad a_-(s, t) = a_O(s, t).$$

These amplitudes (in the (s, t) -representation) are

$$a_{\pm}(s, t) = m_{\pm} \tilde{s}^{\alpha_{\pm}(t)} e^{b_{\pm} t}$$

for the monopole. In this equation,

$$\tilde{s} = \frac{s}{s_0} e^{-i\pi/2} \quad (s_0 = 1 \text{ GeV}^2)$$

to respect s - u crossing, and linear trajectories $\alpha_{\pm}(t)$ are considered:

$$\alpha_{\pm}(t) = 1 + \delta_{\pm} + \alpha'_{\pm} t.$$

The well-known unitarity constraints set the Pomeron-Odderon hierarchy:

$$\alpha'_+ > \alpha'_-, \quad \delta_+ > \delta_-, \quad \text{with } \delta_+ > 0.$$

With the above choices, the ‘‘coupling’’ m_+ is real and negative while m_- is purely imaginary. The optical theorem sets the normalization $\sigma_{\text{tot}} = (4\pi/s) \Im m A(s, t = 0)$.

The corresponding amplitudes h_{\pm} (in the (s, b) -representation) are readily obtained by the Fourier–Bessel transform (3), and they are

$$h_{\pm}(s, b) = \frac{m_{\pm} \tilde{s}^{\alpha_{\pm}(0)}}{4sM_{\pm}} e^{-b^2/4M_{\pm}},$$

with

$$M_{\pm} = \alpha'_{\pm} \ln \tilde{s} + b_{\pm}.$$

It is easy to show that, for large s values and $0 \leq b^2 < R^2$,

$$|e^{2i\lambda_+ h_+}| \approx \exp\left(\frac{m_+ \lambda_+ \cos\left(\frac{\pi}{2} \delta_+\right)}{2s_0 \alpha'_+ \ln s} \left(\frac{s}{s_0}\right)^{\delta_+}\right).$$

Due to the negative sign of m_+ and to the positive sign of δ_+ , this quantity goes to zero when $s \rightarrow \infty$ (for both processes, $\bar{p}p$ and pp).

Similarly, for large s values and $0 \leq b^2 < R^2$,

$$\begin{aligned} & |e^{\pm 2i\lambda_- h_-(1-z)}| \\ & \approx \exp\left(\pm \frac{\tilde{m}_-(\lambda_+ \lambda_- - \lambda_0^2)}{2s_0 \alpha'_- \lambda_+ \ln s} \sin\left(\frac{\pi}{2} \delta_-\right) \left(\frac{s}{s_0}\right)^{\delta_-}\right), \end{aligned}$$

(recall that the sign \pm distinguishes the $\bar{p}p$ and pp scatterings), where we have defined $m_- = i\tilde{m}_-$. One of these quantities diverges if $\delta_- > 0$, irrespective of the sign of $\tilde{m}_-(\lambda_+ \lambda_- - \lambda_0^2)$. Thus, δ_- must be negative (see comment (1) Sect. 3.4).

Had we chosen a dipole⁷ instead of a monopole

$$a_{\pm}(s, t) = d_{\pm} \tilde{s}^{\alpha_{\pm}(t)} e^{b_{\pm}(\alpha_{\pm}(t)-1)} (b_{\pm} + \ln \tilde{s})$$

(or a combination of a monopole with a dipole), similar conclusions would follow.

Appendix C Proof of (33)

We start from the definition of $S(x, y)$ (first line of (33)):

$$\begin{aligned} S(x, y) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{x^n y^m}{(n+m)!} \\ &= xy \sum_{q=0}^{\infty} \frac{y^q}{\Gamma(q+3)} \sum_{p=0}^{\infty} \frac{x^p}{(q+3)_p}, \end{aligned}$$

or [12, 14]

⁷ The difference between a monopole and a dipole is essentially that the amplitude of the latter grows with an additional power of $\ln s$.

$$\begin{aligned} S(x, y) &= \frac{xy}{2} \sum_{q=0}^{\infty} \frac{y^q}{(3)_q} {}_1F_1(1; q+3; x) \\ &= \frac{y}{x} \int_0^x dt e^{ty/x} (x-t) {}_1F_1(1; 2; x-t) \\ &= \frac{y}{x} \int_0^x dt (e^x \cdot e^{t(y/x-1)} - e^{ty/x}) \\ &= \left[\frac{x}{y-x} e^y + \frac{y}{x-y} e^x + 1 \right] \\ &= 1 + \frac{xe^y - ye^x}{y-x}. \end{aligned}$$

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